Practical tutorial on cylindrical structure vibro-acoustics part 1 – vibrations

Stephen A. Hambric[1]
Hambric Acoustics, LLC

ABSTRACT

The mathematics which describe the vibroacoustic behavior of cylindrical structures are imposing to say the least. Part 1 of this practical tutorial demystifies cylindrical shell vibration theory by using measured data from actual shells and pipes to explain key concepts. For any shell, you can estimate frequency ranges where shells behave like simple beams and flat plates, greatly simplifying calculations of modes of vibration and mobilities. The key is first calculating the ring frequency – the frequency where membrane waves can propagate fully around the shell circumference. Simple infinite structure theory may then be used to compute mean mobilities for beam, shell, and flat plate behavior. Modes of vibration for a cylinder depend on both longitudinal and circumferential harmonics, or a helical wavenumber. Cremer's simple approximate resonance frequency formula is used to show examples for a large diameter short shell and a small diameter long shell (a pipe). In all cases in this tutorial, measurements and simple estimates agree well, showing that cylindrical shell vibrations may be estimated without difficult math or complex computer models.

1. INTRODUCTION

This is the first of two tutorials on one of the most complex topics in vibroacoustics – cylindrical shell structures. Those of you who have already read papers or books on cylindrical shell vibration theory have seen already how complicated the equations are. Not only are simple physical insights not obvious when working with these equations, but they are quite difficult to implement properly in computer codes (at least they were for me the first time I tried it as a PhD student many years ago).

This tutorial takes a different approach. Of course, I'll show you the basic equations of motion, but rather than discuss them in detail I will show you some examples of structural mobilities, mode shapes, and resonance frequencies. I will then highlight some of the key quantities that govern how shells vibrate and show you some miraculous and simple formulae you can use to approximate that vibration. You'll also learn about the nomenclature of cylindrical shell modes and see when they occur for two types of shells – a large diameter small thickness shell, and a small diameter, thick, long shell (a pipe).

This is the latest in a series of tutorial articles I have written, including the original two tutorials for Acoustics Today magazine [1, 2] and several others at Inter-noise conferences. You can download them from hambricacoustics.com. For even more on vibroacoustics, see [3].

[1]hambricacoustics@gmail.com

2. CYLINDRICAL SHELL BASIC THEORY

Figure 1 shows a schematic of a finite cylindrical shell of half-length L, radius a, and thickness h. The motion is defined in a cylindrical coordinate system, with z pointing along the shell axis, r radially, and ϕ rotationally. The corresponding vibrations in those directions are u, w, and v. We are mainly interested in how to calculate radial motion w since that couples with interior and/or exterior acoustic spaces (more on this in part 2 of the tutorial). However, as we are about to learn calculating radial motion in a shell requires also calculating the in-plane motion.

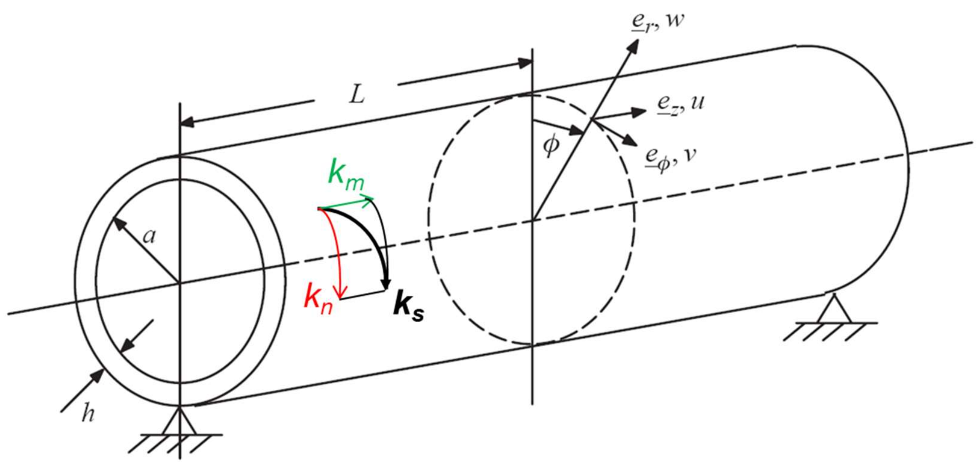

Figure 1: Cylindrical shell dimensions and coordinate system.

Donnell's structural equations of motion[2] at the shell midsurface for a general exterior pressure loading p_a are:

$$\frac{\partial^2 u}{\partial z^2} + \frac{1-\nu}{2a^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{1+\nu}{2a}\frac{\partial^2 v}{\partial z \partial \phi} + \frac{\nu}{a}\frac{\partial w}{\partial z} - \frac{\ddot{u}}{c_p^2} = 0,$$

$$\frac{1+\nu}{2a}\frac{\partial^2 u}{\partial z \partial \phi} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial z^2} + \frac{1}{a^2}\frac{\partial^2 v}{\partial \phi^2} + \frac{1}{a^2}\frac{\partial w}{\partial \phi} - \frac{\ddot{v}}{c_p^2} = 0, \quad \text{and}$$

$$\frac{\nu}{a}\frac{\partial u}{\partial z} + \frac{1}{a^2}\frac{\partial v}{\partial \phi} + \frac{w}{a^2} + \beta^2\left(a^2\frac{\partial^4 w}{\partial z^4} + 2\frac{\partial^4 w}{\partial z^2 \partial \phi^2} + \frac{1}{a^2}\frac{\partial^4 w}{\partial \phi^4}\right) + \frac{\ddot{w}}{c_p^2} - \frac{p_a(1-\nu^2)}{Eh} = 0,$$

where:

ν is Poisson's ratio,

$c_p = \sqrt{E/[(1-\nu^2)\rho_s]}$ is the complex speed of sound in the shell,

$\beta^2 = h^2/12a^2$,

$E = (1+i\eta)E_{real}$ is the complex Young's Modulus,

ρ_s is the structural mass density, and

η is the material loss factor.

---

[2] For details on Donnell's and other formulations (there are several), I highly recommended Leissa's book [4].

I have color coded the vibration directions as:

- red for in-plane axial motion,
- blue for in-plane twisting/torsional motion, and
- green for transverse (radial) motion.

Don't worry, we won't be discussing all the details of this horrifying set of equations. Instead, just notice that:

- the in-plane axial and twisting parts look like simple wave equations with second order spatial derivatives in cylindrical coordinates (see the like-colored boxes), with 'cross-terms' from the other directions (the other colors); and
- the transverse part looks like a flexural/bending wave equation with fourth order spatial derivatives, but again with cross-terms from the other directions.

The cross-terms mean that *in-plane motion is coupled to transverse motion* – this is a key to understanding how curved structures vibrate. Consider what happens if you pull on the shell in Figure 1 axially. Not only will the shell deform in the axial direction, but the Poisson effect of the material will cause it to pull inward. Push on the shell and it will bulge outward. The same thing happens if you twist the shell – the walls will deflect radially. These effects are ignored for flat plates, but cannot be for curved shells.

In other articles and books on shell theory, more painful derivations follow, all leading to complex series of equations spanning several pages which allow you to estimate vibrations and radiated sound (if you implement them properly that is). These are not for the faint of heart and can confound even seasoned PhDs in vibroacoustics. This tutorial, however, is intended to explain shell theory through simple examples, and give you just enough background and insight to approximate vibrations and modes for any shell.

3. MOBILITIES

Cylindrical shell geometries are commonly defined by ratios like length to radius (L/a) and thickness to radius (h/a). Ben Doty in his MS thesis [5] and NoiseCon articles [6,7] describes a series of in-air and in-water measurements he made on two cylindrical shell steel structures shown in Figure 2 and Table 1 – one a large diameter, small thickness shell; and the other a small diameter, large thickness shell (a pipe). Ben's pipe has the added complication of an elbow, which makes it 'doubly curved'. We won't focus on this aspect here though.

Let's examine some measured drive point mobilities (radial velocity response to a radial applied force) for the two shells, starting with the pipe in Figure 3. There are several clear modal peaks in the measured mobility (more on these in the next section). Two other curves are plotted with the mobility – estimates for an infinite beam and infinite shell structure with the same cross-sectional and material properties. A measured drive point mobility of the large shell is shown in Figure 4. More modes appear in this mobility than in the beam's (since the structure is larger). Here, the two infinite structure curves are of a shell and a flat plate (the infinite beam mobility is not included). Not only do the infinite structure mobilities approximate the means of the measured mobilities for both structures over different frequency ranges[3]; they also provide useful insights into how shell structures behave over those ranges.

---

[3] For more on the Amazing Uses of Infinite Structure Theory, see [8].

Table 1: Shell structure dimensions for the large diameter steel shell (top) and pipe (bottom).

L (m)	a (m)	h (mm)
1.20	0.460	9.5
1.04	0.044	5.5

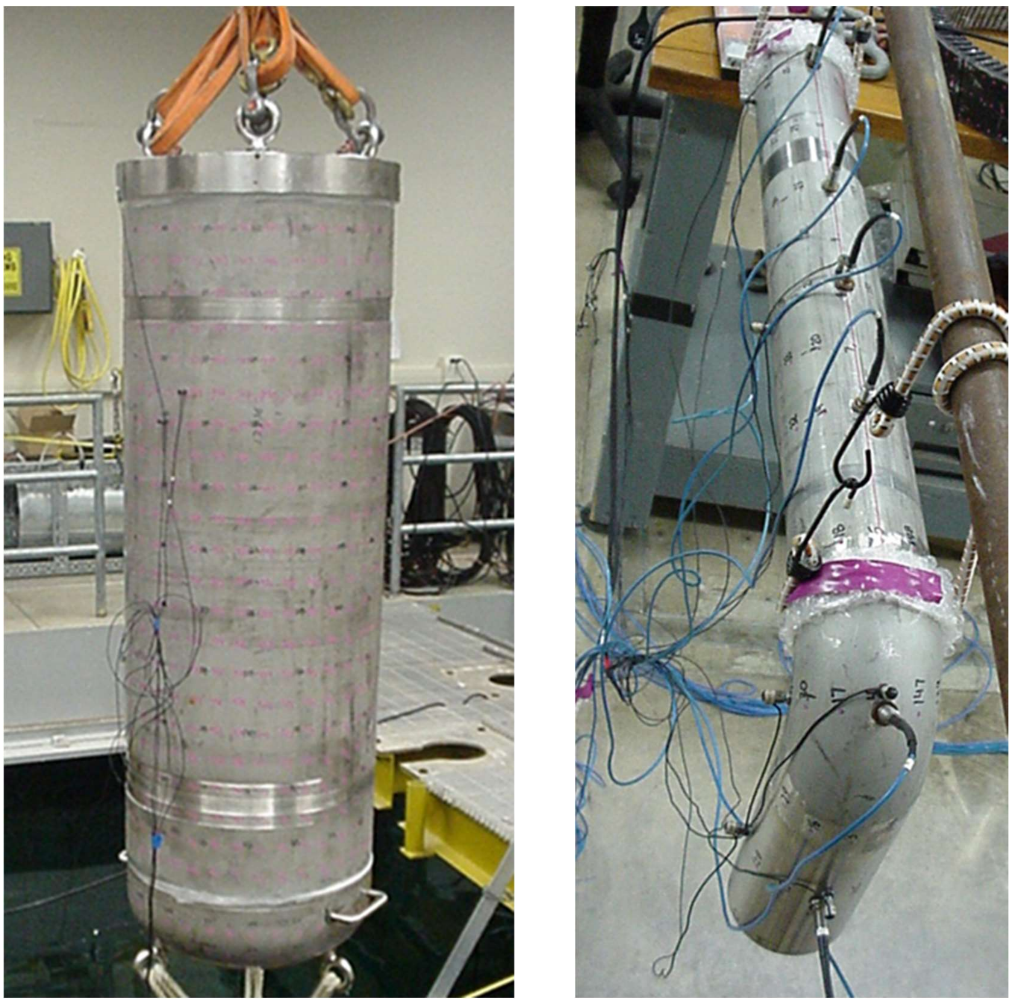

Figure 2: Large diameter steel shell with $L/a = 5.33$ and $h/a = 0.0417$ (left) and small diameter steel shell with $L/a = 23.4$ and $h/a = 0.123$ (right). Both are instrumented with accelerometers and the small diameter shell (or pipe) is instrumented with flush mounted pressure taps to measure internal acoustic pressures.

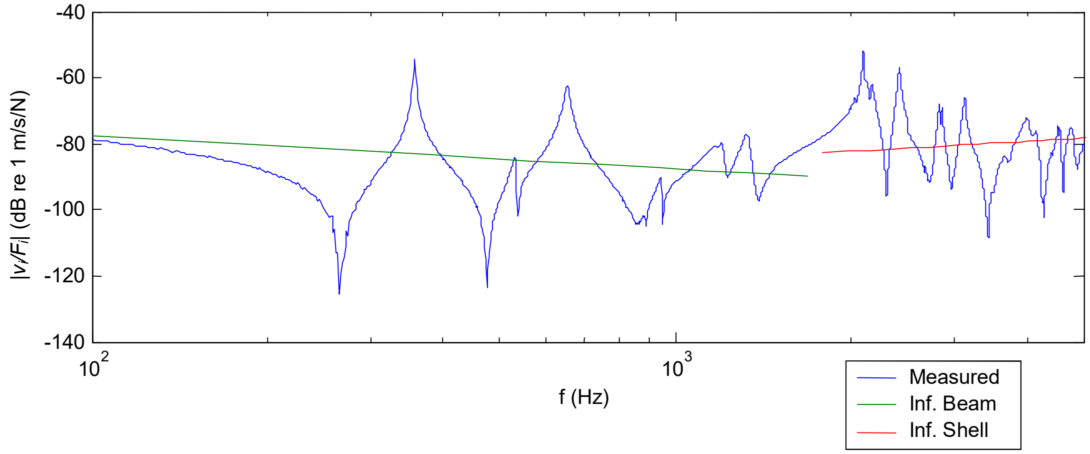

Figure 3: Sample drive point mobility of pipe.

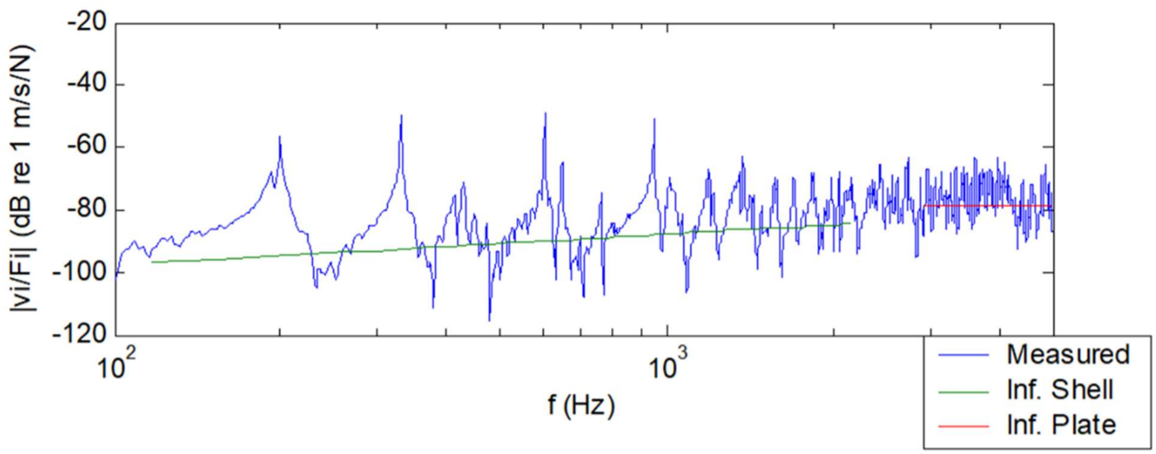

Figure 4: Sample drive point mobility of large shell.

The infinite beam, shell, and flat plate mobilities are:

Beam: $$Y = \left[4\pi a \rho_s h \sqrt{\frac{\Omega c_l^2}{\sqrt{2}}}\right]^{-1} \qquad \Omega < 0.77\frac{h}{a}$$

Shell: $$Y = \frac{0.66}{2.3 c_l \rho_s h^2}\sqrt{\Omega} \qquad 0.77\frac{h}{a} < \Omega < 0.6$$

Plate: $$Y = \frac{1}{8\sqrt{D\rho h}}, \qquad \Omega > 0.6$$

The beam and shell equations introduce a new variable: Ω. This is a critical parameter in shell theory, and is the ratio of frequency to *ring frequency*, which is the frequency at which the in-plane wavelength matches the shell circumference:

$$f_{ring} = \frac{c_p}{2\pi a}$$

Frequency is normalized as

$$\Omega = \frac{f}{f_{ring}} = \frac{\omega}{\omega_{ring}}$$

Note the approximate frequency ranges of validity for the infinite structure equations. Using typical material properties of steel and the dimensions in Table 1 we compute a ring frequency for the pipe of 18.8 kHz. The mean mobility for the pipe therefore transitions from that of a beam to a shell at about 0.77 $(h/a)\,f_{ring}$, or 1780 Hz. The ring frequency of the large shell, 3.65 kHz, is much lower (since the radius is much larger). The mean mobility for the large shell therefore transitions from that of an infinite curved shell to that of a flat plate at 0.6 f_{ring}, or 2190 Hz. To understand the physical reasons for these frequency ranges of behavior and the transition frequencies, we need to examine the shell modes of vibration.

4. MODES OF VIBRATION

Modes of a structure are just the eigensolutions of the equations of motion. Analytic expressions are available for resonance frequencies and mode shapes of simple beams and flat plates with idealized boundary conditions. However, the cylindrical shell equations of motion are not readily adapted to simple mode shape and frequency equations. Fortunately, Cremer, Heckl, and Ungar [9] derived a reasonable approximation of modal frequencies for radial motion of cylindrical shells:

$$\Omega_{m,n}^2 = \frac{\omega_{m,n}^2}{\omega_{ring}^2} \cong \frac{k_m^4}{k_s^4} + \beta^2\left[k_s^4 a^4 - \frac{n^2(4-\nu)-2-\nu}{2(1-\nu)}\right]$$

Note once again the normalization by the ring frequency. This formula is general and may be used for any combination of boundary conditions on the shell ends. The boundary conditions determine the modal wavenumber k_m for axial harmonics m. k_s is the amplitude of the two-dimensional wavevector (shown in Figure 1 on the shell):

$$k_s = \sqrt{k_m^2 + k_n^2}$$

where k_n is the wavenumber in the circumferential direction, and is always equal to n/a, where n is the circumferential harmonic.

Figure 5 shows the low order modes of a simply supported cylindrical shell for the first three axial and circumferential harmonics. The shapes are:

$$\varphi_{mn} = A_{mn}\cos(k_m z)\cos(n\phi) \text{ and } A_{mn}\cos(k_m z)\sin(n\phi)$$

where k_m is $m\pi/2L$ for simple end supports. Look at the n=0 mode shapes first. The entire circumference moves in phase – these are known as the 'breathing' or 'ring' modes, and generally resonate at frequencies very close to the ring frequency. Now consider the n=1 mode shapes. Here, the entire circumference vibrates as a rigid body. The motion resembles that of a simple beam with a circular cross section. This is why pipe mobilities resemble simple beam mobilities at low frequencies! Finally, examine the n=2, or 'lobar' mode shapes. The cross section 'ovals' as it oscillates. These are the lowest order shell modes. As n increases there are more waves around the circumference. The mode shapes in Figure 5 only show the $\cos(n\phi)$ modes. For non-breathing modes ($n > 0$) there are two instances of each mode (cos and sin).

Now let's exercise Cremer, Heckl, and Ungar's approximate resonance frequency formula, starting with a high L/a and small h/a shell. Figure 6 shows approximate resonance frequencies, normalized by the ring frequency of course, for the first four axial harmonics m and 14 circumferential harmonics n. First, notice that resonance frequencies increase with increasing m – this is consistent with resonances of flat plates and beams. However, now consider how the resonance frequencies vary with increasing n. The resonance frequencies at lower harmonics are quite high, and at $n = 0$ cluster around the ring frequency. This is the stiffening effect of the in-plane membrane waves. At much higher circumferential harmonics, though, the resonance frequencies increase with increasing n, consistent with flat plates.

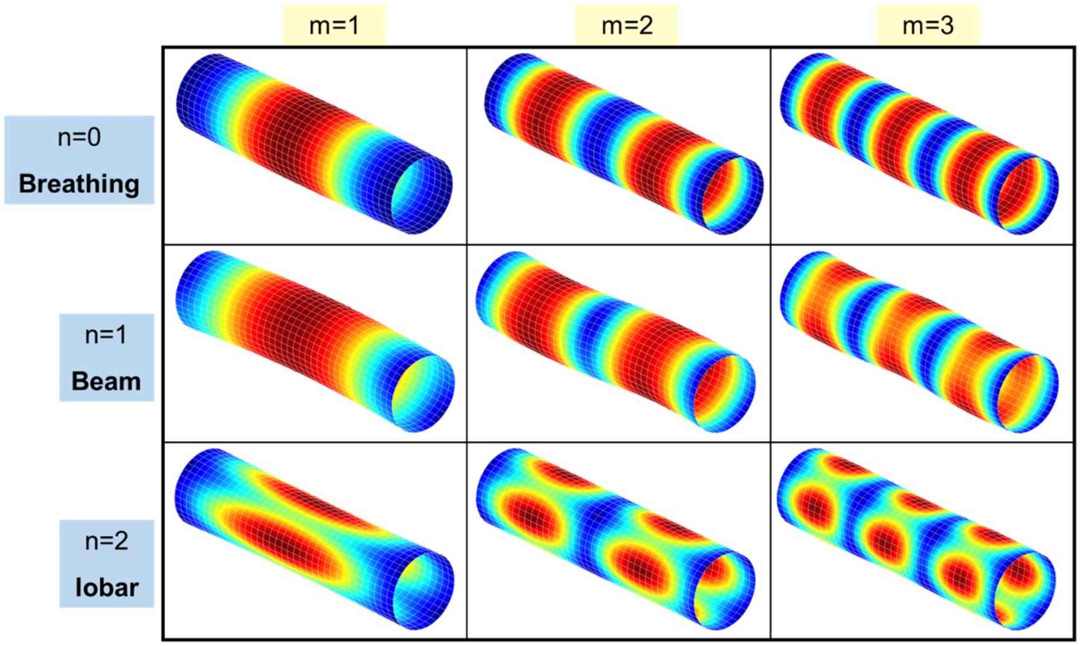

Figure 5: Low order radial modes of vibration of a cylindrical shell.

Now, consider cutting and unrolling the shell into a flat rectangular plate and adding simple supports along the cut edge. Resonance frequencies for a simply supported plate are:

$$\omega_{mn} = \sqrt{\frac{D}{\rho h}} \left[k_m^2 + k_n^2 \right]$$

The plot on the bottom of Figure 6 adds the m=1 and m=4 resonance frequencies for the unrolled flat plate. Notice how they are nearly identical to the frequencies of the shell at very high circumferential harmonics. This shows that the membrane effects diminish with increasing n. As more waves wrap around the circumference they become flatter, and eventually behave as they do in infinite flat plates. *This is why the high frequency mobility of a shell resembles that of a flat plate, as we see in Figure 4*. This phenomenon applies to any curved structure – very high frequency waves (with very short wavelengths with respect to curvature) are essentially the same as those in flat infinite plates.

Figure 7 shows resonance frequencies for a long thick shell – a pipe, with L/a = 60 and h/a = 0.1. The lowest frequency modes are all n=1, or beam-like bending modes. The next group of frequencies correspond to the n=2 ovaling modes – these are almost always the lowest frequency shell modes in a pipe. *Now we can better understand the pipe mobility in Figure 3 – it is dominated by n=1/beam effects at low frequencies, and transitions to shell behavior when the n=2 modes cut on*.

So, when working with a new cylindrical shell, you should always:

- first compute the ring frequency;
- compute the different transition frequencies for infinite structure behavior – these will partition the frequency ranges of beam-like behavior (low frequency), shell behavior (mid frequency) and flat plate behavior (high frequency); and
- estimate mean mobilities using the infinite structure formulae.

This will give you simple and fast estimates of the mobility of any cylindrical shell.

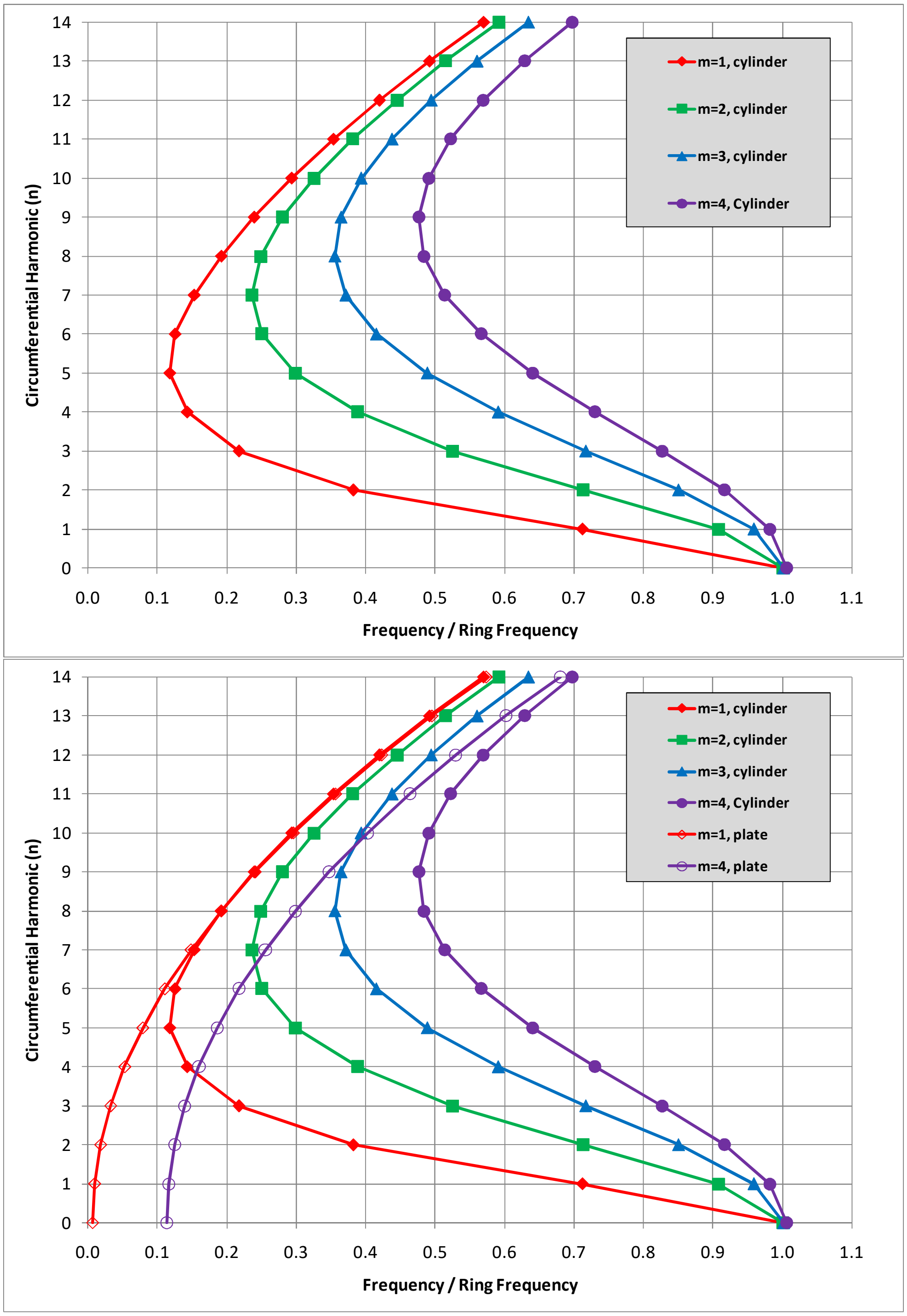

Figure 6: Non-dimensional resonance frequencies of a 2 m diameter, 1 m long ($L/a = 0.5$), 10 mm thick ($h/a = .01$) steel cylinder with simply supported ends. Top – shell only, Bottom – shell compared to equivalent flat simply supported plate.

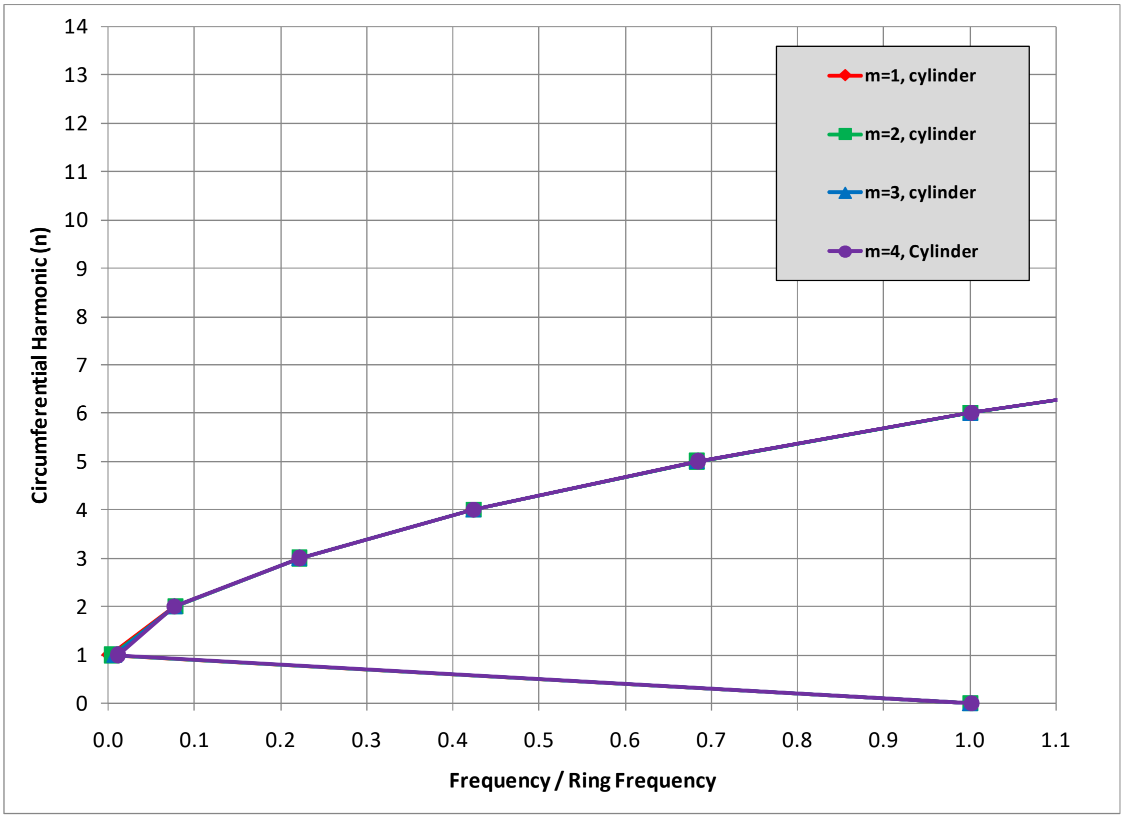

Figure 7: Non-dimensional resonance frequencies of a simply supported pipe L/a=60 and h/a=0.1.

5. SUMMARY AND CONCLUSIONS

The goal of this tutorial was to uncomplicate the vibrational behavior of cylindrical shells. You have seen simple examples of mobilities and modes of vibration, along with equations to approximate cylindrical shell vibrations. Once again, the first key quantity to compute when confronted with any cylindrical shell problem is the ring frequency. All behavior is grounded on that quantity. Small diameter thick-walled long shells are essentially beams at low frequencies. You can estimate the frequency when shell effects become important using the transition frequency shown in Section 3. Large diameter thin-walled shells behave like flat plates at high frequencies, when wavelengths become small with respect to the shell curvature. Once again, use the transition frequency in this paper to estimate when this occurs.

If you find the equations of shell motion horrifying, then the equations that define sound radiated within or outside shells are even more so. Part 2 of this tutorial will provide a simpler, gentler approach to estimate and understand cylindrical shell sound-structure interaction.

6. REFERENCES

1. S. Hambric, "Structural acoustics tutorial - part 1: Vibrations in structures," *Acoustics Today*, Vol. 2, No. 2, 2006.
2. S. Hambric and J. Fahnline, "Structural acoustics tutorial - part 2: Sound-structure interaction," *Acoustics Today*, Vol. 3, April 2007.
3. S. Hambric, D. Nefske, and S. Sung, *Engineering Vibroacoustic Analysis: Methods and Applications*, Wiley, 2016.
4. Leissa, A., *Vibration of Shells*, NASA SP-160, 1969.

5. Doty, B.J., *Structural-acoustic measurements of a large, thin-walled cylindrical shell and a small thick-walled pipe with a ninety degree elbow, under water loading*, Penn State MS thesis, 2004.
6. Doty, B.J., et. al., "Structural-acoustic measurements of a submerged cylindrical shell," *Proceedings of Noise-Con 2004*, pp. 183-192, Baltimore, MD, July 2004.
7. Doty, B.J., et. al., "Structural-acoustic measurements of pipes with ninety-degree elbows, under water loading," *Proceedings of Noise-Con 2005*, Minneapolis, MN, October 2005.
8. S. Hambric, "To infinity and beyond: the amazing uses of infinite structure mobility theory," in *Proceedings of Internoise 2019*, Madrid, Spain, June 2019.
9. Cremer, L, Heckl, M., and Ungar, E., Structure-borne sound, second edition, Springer Verlag, 1988.